Announcements

1) Colloquium tomorrow,

3-4, CB 2070

On plasma modelling

2) New assignment up,

due next Tuesday

Definition: If $A, B \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$), then A and B are **similar** if $\exists$ an invertible $S \in M_n(\mathbb{C})$ (or $M_n(\mathbb{R})$) with

$$\boxed{B = SAS^{-1}}$$

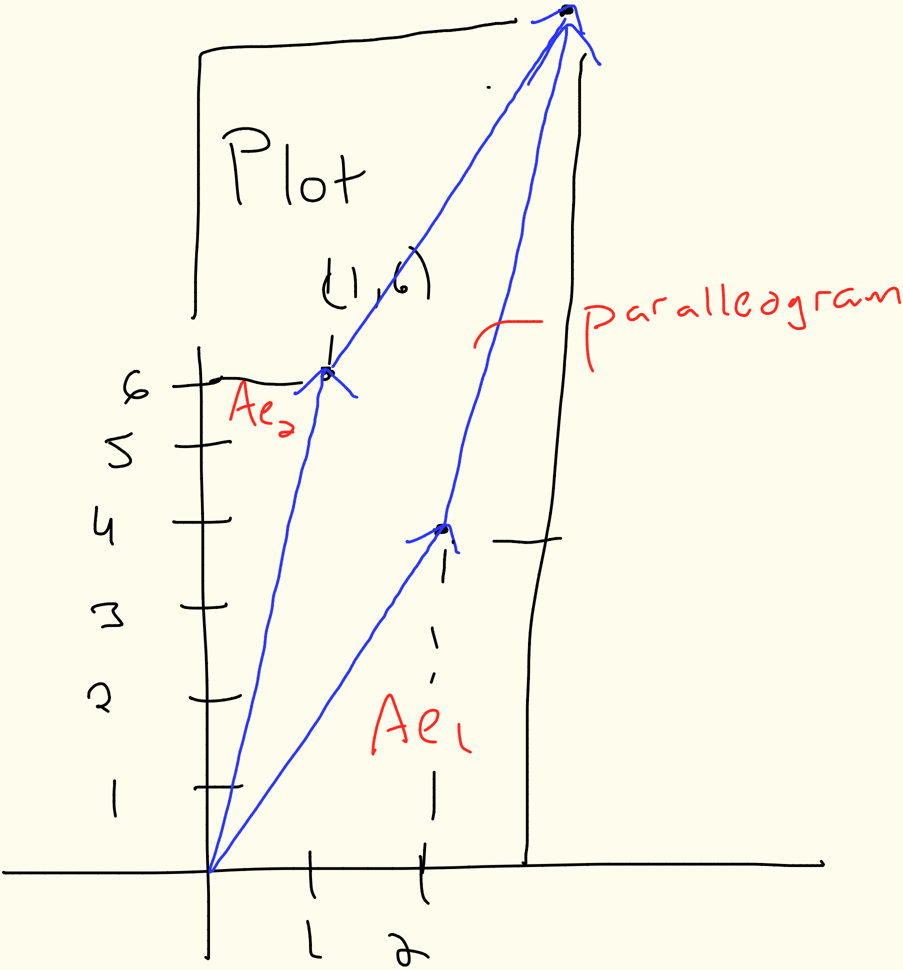
Example 1: (2x2 determinant, geometry)

Let $A = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix}$.

Then A is invertible,

and $$Ae_1 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$Ae_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$


Plot
(1,6)
parallelogram
Ae_2
Ae_1
6
5
4
3
2
1
1
2

Find area of box,

Subtract the area of the triangles and resulting rectangles to get the area of the parallelogram

When you do, you'll find

$\boxed{\text{Area} = 8}$

This is exactly the determinant of A, where we define

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 6 \end{bmatrix},$$

so $\det(A) = 8$.

In general, if $A \in M_2(\mathbb{R})$ (or $M_2(\mathbb{C})$) is invertible,

then

$$\boxed{|\det(A)| = \text{area of parallelogram spanned by } Ae_1 \text{ and } Ae_2}$$

If A is not invertible, then Ae_1 and Ae_2 are scalar multiples, and so do not form a parallelogram! This gives a figure with zero area, again equal to $\det(A)$.

Recall: (S_n) If X is a set with n elements, S_n denotes all the bijections from X to itself. The composition of any two such bijections is again a bijection of X.

We can consider

$$X = \{1, 2, \cdots, n\}$$

and S_n as the bijections on this set.

Definition: (transitivity)

Let $Y \subseteq X$. Then

$\sigma \in S_n$ is **transitive**

on Y if for any

two elements $i, j \in Y$,

$\exists$ $k \in \mathbb{N}$ with

$$\boxed{\sigma^k(i) = j}$$

($\sigma^k = \sigma$ composed k times w/ itself)

Example 2: (S_4)

Let $\sigma : \{1,2,3,4\} \to \{1,2,3,4\}$

$\sigma(1) = 2, \quad \sigma(2) = 1$

$\sigma(3) = 4, \quad \sigma(4) = 3.$

Then σ is transitive on

$\{1,2\}$ since

$\sigma(1) = 2, \quad \sigma(\sigma(1)) = 1$

$\sigma(2) = 1, \quad \sigma(\sigma(2)) = 2$

σ is also transitive on $\{3,4\}$, but not on $\{1,2,3\}$ since no power k will give

$$\sigma^k(1) = 3.$$

Now if

$$\tau : \{1,2,3,4\} \to \{1,2,3,4\}$$

$$\tau(1)=2, \quad \tau(2)=3$$

$$\tau(3)=4, \quad \tau(4)=1,$$

then τ is transitive on $\{1,2,3,4\}$.

Remark: If σ is transitive on $Y \subseteq X$, then σ is transitive $\forall$ $Z \subseteq Y$.

Notation: (cycle)

If $\sigma : \{1,2,3\} \to \{1,2,3\}$

$\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$,

then σ is transitive on

$\{1,2,3\}$. We shorthand

$$\sigma = (1\ 2\ 3)$$

Let $\gamma \in S_6$.

If we write

$$\gamma = (4\,3\,1)(2\,5)(6)$$

this means

$\gamma(4) = 3$, $\gamma(3) = 1$, $\gamma(1) = 4$

$\gamma(2) = 5$, $\gamma(5) = 2$

$\gamma(6) = 6$

In general, if

$\sigma \in S_n$, σ is transitive

on $\{1, 2, \ldots, n\}$, write

$$\sigma = \left(1 \ \sigma(1) \ \sigma^2(1) \ \sigma^3(1) \ldots \sigma^{(n-1)}(1)\right)$$

and call σ a cycle.

If σ is any bijection on $X=\{1,2,\ldots,n\}$, then we can decompose X as $X=\bigsqcup_{i=1}^{k} X_i$

where $k\leq n$ and σ is transitive on X_i but not transitive on any larger set containing X_i $\forall$ $1\leq i\leq k$.

Example 3:

If $\gamma = (431)(25)(6)$,

then $X_1 = \{1, 3, 4\}$

$X_2 = \{2, 5\}$

and

$X_3 = \{6\}$.

Given $\sigma \in S_n$ and breaking X up into $\bigsqcup_{i=1}^{k} X_i$ as indicated, then writing σ as the product of cycles on the X_i's gives the cycle decomposition of σ.

Example 4:

If $\sigma \in S_9$,

$X_1 = \{1, 2, 5, 9\}$

$X_2 = \{8, 7\}$

$X_3 = \{3, 4\}$

$X_4 = \{6\}$, then we

write

$\sigma = (6)(34)(87)\sigma_1$

where σ_1 is the cycle of σ on $\{1, 2, 5, 9\}$.

If $\sigma(1) = 2$, $\sigma(2) = 9$
$\sigma(9) = 5$, $\sigma(5) = 1$,

$$\sigma_1 = (1\ 2\ 9\ 5)$$

But this can change depending on σ!

Definition: (transposition)

$\sigma \in S_n$ is called a **transposition** if

$\exists$ i, j , $1 \leq i, j \leq n$,

$i \neq j$,

$$\boxed{\sigma(i) = j \, , \; \sigma(j) = i}$$

and $\sigma(k) = k$ $\forall$

$k \neq i, j$, $1 \leq k \leq n$.

Theorem: (cycle decomposition)

Writing $\sigma \in S_n$ as the product of disjoint cycles, we can write each cycle as a product of transpositions.

The total number of transpositions in σ can only be either always odd or always even.

proof: Math 412!

$\square$

Example 5:

$\sigma \in S_9$

$$\sigma = (1\ 2\ 5\ 9)(3\ 4\ 8)(6\,7)$$

$$(1\,2\,5\,9) = (1\,2)(2\,5)(5\,9)$$

$$(3\,4\,8) = (3\,4)(4\,8)$$

$$(6\,7) = (6\,7).$$

$$\boxed{\sigma = (1\,2)\,(2\,5)(5\,9)\,(3\,4)\,(4\,8)\,(6\,7)}$$